Recitation 9: Martingale Convergence Theorems

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Exercise 1 (Question 1 of course)**.** *Recall the definition of U.I. condition. Prove two practical conditions to prove U.I. condition for* $(X_i)_{i\in I}$*:*

1. *There exists* $p > 1$*, such that* $\sup_i \mathbb{E}[|X_i|^p] < \infty$.
2. *There exists* $Z \in L^1$*,* $Z \geqslant 0$*, such that for every* X_i*,* $|X_i| \leqslant Z$.

Exercise 2 (Question 2 of course)**.** *Recall the definition of backward martingale and use it to prove the law of large number of version* L^1.

Exercise 3. *Prove the following theorem: A sequence* X_n *converges to* X *in* L^1 *if and only if it converges in probability to* X *and it is uniformly integrable.*

Exercise 4. *Let* $(X_i)_{i\in\mathbb{N}}$ *be i.i.d. random variables such that* $\mathbb{E}[X_1] = 0$ *and* $\operatorname{Var}[X_1] = \sigma^2 < \infty$. *Let* $S_n = \sum_{i=1}^n X_i$ *and prove that for* $x > 0$

$$\mathbb{P}[\max_{1\leqslant i\leqslant n} S_i > x] \leqslant \frac{1}{x^2}\operatorname{Var}[S_n].$$

Exercise 5 (Galton-Watson process)**.** *Let* $(X_{n,i})_{n,i\in\mathbb{N}}$ *be i.i.d. random variables taking value in* $\mathbb{N}$ *with finite expectation*

$$\mathbb{E}[X_{1,1}] = m < \infty,$$

and $(Z_n)_{n\in\mathbb{N}}$ *be a Galton-Watson process defined by*

$$Z_0 = 1, \qquad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n+1,i}.$$

We also define its natural filtration $\mathcal{F}_n = \sigma((X_{j,i})_{j\leqslant n, i\in\mathbb{N}})$ *and the time of extinction*

$$\tau = \inf\{n | Z_n = 0\}.$$

1. *Prove that* τ *is a stopping time with respect to* $(\mathcal{F}_n)_{n\in\mathbb{N}}$.
2. *Prove that if* $m < 1$*, then* $\tau < \infty$ *almost surely.*
3. *Prove that* $M_n := \frac{Z_n}{m^n}$ *is a martingale with respect to* $(\mathcal{F}_n)_{n\in\mathbb{N}}$*, and converges almost surely to a limit* M_∞.
4. *What is the limit for the case* $m < 1$ *? Does* M_n *converge in* L^1 *to* M_∞ *?*
5. *Suppose that* $\operatorname{Var}[X_{i,i}] = \sigma^2 < \infty$*, prove that for the case* $m > 1$*,* $M_n \xrightarrow{L^2} M_\infty$. *Then deduce* $\mathbb{P}[\tau < \infty] < 1$.